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Axial vacuum symmetry of the unified gauge theories with the gravitational mechanism of instability

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Abstract. The problem of stable-state determination is considered in unified gauge theories involving gravitation. The gravitational fields are examined at a classical level. The self-consistent set of field equations is studied in the semiclassical approach. A new determination of stable states is given. If the conditions of the determination are not satisfied then the vacuum state may be metastable. For the case of axial symmetry the metastable Z-type states are shown to cancel if these states are symmetric.

1. Introduction

The theory of gauge fields led to the construction of the unified theory of weak and electromagnetic interactions (Weinberg 1967, Salam 1968). The possibility of unification of the strong interaction is discussed. At the same time a gauge approach to gravitation is known. One may hope, therefore, that the gauge treatment of gravitation (Kibble 1961) will allow us to include this interaction in the framework of the general scheme.

In the present work we use the classical gauge theory of gravitation to examine the problem of how gravitational fields, which are considered here at a classical level, can provide stability of the vacuum states in the unified gauge theories.

It is well known that a state is stable if the time of its existence is infinite. When a time of existence t is finite a vacuum state is called metastable. At $t \to 0$ a state is unstable. In the framework of the unified gauge theories a symmetric vacuum state must be unstable since the massless fields become massive only in the asymmetric vacuum states. If, for instance, a symmetric state is metastable the mass spectrum is not formed during the finite time of its existence.

This fact may occur when the interaction of quantum fields is described in a curved space–time. Unlike previous works (Domokos 1976, Grib and Mostepanenko 1977) our assumption about the consistency of gravitational fields with quantum fields by means of the semiclassical field equations is here principal.

As has been shown in earlier works (Nikolaenko 1977a, b, 1980) the stable vacuum state definition in the presence of gravitation is transformed to the following definition.

In systems with spontaneous symmetry breaking, where the mechanism of instability is realised by taking into account the classical gravitational fields, the vacuum state is stable when: (i) the vacuum is determined by the minimum of the Higgs potential or (ii) there exists a gravitational topological charge which is conserved.

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If both these conditions are not fulfilled, then the vacuum may be metastable. In the self-consistent approach the problem arises as to which type of gravitational field symmetry corresponds to the metastable vacuum states. In the present work we consider the axially symmetric gravitational fields when the axis of symmetry Z exists in each point of a curved space–time. One may assume that the existence of the Z axis specifies a vacuum state as a metastable one. We call this Z-type metastability.

The purpose of this paper is to prove the following proposition.

When the vacuum state is symmetric the Z-type metastability is cancelled in the self-consistent unified gauge theories with gravitation.

The proof of this proposition without loss of generality will be related to the SU(2) ℝ U(1)-gauge invariant unified theory of Weinberg and Salam.

2. The Lagrangian and the field equations

It is known that introduction of the imaginary bare Higgs mass is unnecessary when gravitation is taken into account. In this approach Higgs' field is not massive but conformally invariant (Domokos 1976, Grib and Mostepanenko 1977). Unlike the traditional theory, where the effective mass of Higgs field $M_{\rm H}$ is a free parameter, the mass $M_{\rm H}$ is here defined (Nikolaenko *et al* 1981).

In our version the complete Lagrangian includes the gravitational part $\mathscr{L}_{\rm g}$. This Lagrangian, as Lagrangians for other gauge fields, must be quadratic in strength. Since the curvature tensor may be interpreted as a strength of the gravitational field the part $\mathscr{L}_{\rm g}$ is represented by the following linear functional

$$\mathscr{L}_{\rm g} = \mathscr{L}_{\rm g}[\mathscr{R}^2] \tag{2.1}$$

where $\mathscr{R}^2$ denotes a linear combination of the squares of the curvature tensor and its contractions.

In general the Lagrangian (2.1) is not invariant with respect to the conformal transformations:

$$\mathrm{d}S'^2 = \Lambda^2(x)\,\mathrm{d}S^2. \tag{2.2}$$

It is easy to show at the same time that (2.1) is invariant under the Weyl group combining the conformal transformations (2.2) and the gauge transformations:

$$\gamma'(x) = \gamma(x). \tag{2.3}$$

Here $\gamma(x)$ is a connection one-form. The coefficients of the form $\gamma(x)$ in the case of the Riemannian connection are the Christoffel symbols Γ^i_{jk}.

To prove the invariance of the quadratic Lagrangian under the Weyl transformations (2.2) and (2.3) let us consider the quadratic terms $R^i_{klm}R_i^{klm}$, $R_{ik}R^{ik}$ and R^2. Actually, using the formula

$$R^i_{kjm} = \frac{\partial\gamma^i_{km}}{\partial x^l} - \frac{\partial\gamma^i_{kj}}{\partial x^m} + \gamma^i_{aj}\gamma^a_{km} - \gamma^i_{am}\gamma^a_{kj} \tag{2.4}$$

one obtains

$$\begin{aligned} R'^i_{klm} &= R^i_{klm} \qquad & R' &= \Lambda^{-2}R \\ R'_{ik} &= R_{ik} \qquad & \sqrt{-g'} &= \Lambda^4\sqrt{-g}. \end{aligned} \tag{2.5}$$

It follows that

$$\sqrt{-g'}R'^{i}_{\ klm}R'^{klm}_{i} = \sqrt{-g}R^{i}_{\ klm}R^{klm}_{i}$$

$$\sqrt{-g'}R'_{ik}R'^{ik} = \sqrt{-g}R_{ik}R^{ik}$$

$$\sqrt{-g'}R'^{2} = \sqrt{-g}R^{2}.$$

We take as our Lagrangian for the Higgs field

$$\mathscr{L}_{\mathrm{H}} = \sqrt{-g}[-\nabla^{a}\varphi^{*}\nabla_{a}\varphi + \tfrac{1}{6}R\varphi^{*}\varphi - \lambda(\varphi^{*}\varphi)^{2}]. \tag{2.6}$$

Here the scalar curvature $R>0$ and $R = g^{km}R^{i}_{\ kim}$. The signature of the metric is chosen as $(+++-)$. The self-action constant $\lambda > 0$ and the scalar field

$$\varphi = \sqrt{\tfrac{1}{2}}(\varphi_1 + \mathrm{i}\varphi_2).$$

As in the Weinberg–Salam model, the derivative ∇ is invariant under the group of phase transformations U(1). When ∇ acts on vectors or tensors this operator is invariant, simultaneously, under the group of coordinate transformations. The field φ is conformally transformed under (2.2) as follows

$$\varphi' = \Lambda^{-1}\varphi. \tag{2.7}$$

When the connection is Riemannian, i.e. $\gamma^{i}_{jk} = \Gamma^{i}_{jk}$, the Higgs field equation corresponding to the Lagrangian (2.6) is invariant under the group of conformal transformations (Penrose 1964, Chernikov and Tagirov 1968). It should be emphasised that the Higgs field (2.6) for the Riemannian connection is also invariant under the Weyl transformations restricted by the condition $\Lambda = \text{constant}$, the so-called scale transformations. But the gravitational part $\mathscr{L}_{g}$ is invariant with respect to the complete Weyl group.

From (2.6) it follows that the minimum of the Higgs field potential is given by

$$\eta^{2} = R/12\lambda \tag{2.8}$$

where

$$\eta = \langle\varphi\rangle = \text{constant}. \tag{2.9}$$

Quantisation of the Higgs field near its minimum gives the effective Higgs mass M_{H} not equal to zero and

$$M_{\mathrm{H}} = 2\eta\sqrt{\lambda} = \sqrt{R/3}. \tag{2.10}$$

Relations (2.8) and (2.9) mean the spontaneous breaking of gauge symmetry. In our version this symmetry breaking involves the breaking of conformal symmetry. But the scale invariance remains as $R = \text{constant}$ and $\Lambda = \text{constant}$ according to (2.8) and (2.9).

The first-order variational formalism is used, where the metric components g_{ik} and the connection coefficients γ^{i}_{jk} are considered as two sets of independent variables. Unlike the second-order variational formalism the gravitational field equations in this case form two sets of second-order differential equations in g_{ik} and γ^{i}_{jk}.

At the quantum level this theory contains not only gravitons but particles corresponding to the torsion field (Hehl *et al* 1976, Popov 1976). One may hope, therefore, that this theory avoids the well known difficulty with 'ghosts' (Nouri-Moghadam and Taylor 1976).

Now the gravitational fields are considered at a classical level and the field equations as in Melnikov and Orlov (1979) have the form

$$\delta\mathscr{L}_g/\delta g_{ik} = \kappa\langle T_{ik}^{(\mathrm{ext})}\rangle_\eta \tag{2.11}$$

where κ is the gravitational Einstein constant and $\langle T_{ik}^{(\mathrm{ext})}\rangle_\eta$ is the vacuum expectation value for the energy momentum tensor of quantum fields. The values are averaged in the degenerate vacuum state η. The right-hand side of equation (2.11) is defined by the single Higgs field (2.6), since the vacuum expectation values for the other quantum fields are equal to zero.

The field equation (2.11) is considered as the equation for the asymmetric vacuum state, where conditions (2.8) and (2.9) are fulfilled. In particular, it follows that the scalar curvature

$$R = \text{constant}. \tag{2.12}$$

As has been shown earlier (Nikolaenko 1980) it follows from (2.12) that the effective Lagrangian $\mathscr{L}_g^{(\mathrm{eff})}$ is

$$\mathscr{L}_g^{(\mathrm{eff})} = \mathscr{L}_\mathrm{E} + \mathscr{L}_g[\mathscr{R}^2] \tag{2.13}$$

where $\mathscr{L}_\mathrm{E} = \sqrt{-g}R$, and $\mathscr{L}_g[\mathscr{R}^2]$ is defined as before. It is significant that our version, unlike the purely quadratic gravitational theories, involves the massive sources of the gravitational field.

Since the gravitational fields here are considered at a classical level we examine the field equations in the limit of vanishing torsion, i.e. the torsion form

$$\Sigma(x) = 0.$$

At the same time this restriction may be explained by the fact that the field equations will be studied in the symmetric vacuum state, where it is natural to assume the most simple space–time geometry.

The torsion form $\Sigma(x)$ is known as

$$\Sigma = D\theta$$

where D is a complete (covariant) derivative and $\theta(x)$ is a field of frames. Thus, our restriction may be interpreted as a gauge given by the following relation

$$D\theta = 0. \tag{2.14}$$

The constraints

$$\delta\mathscr{L}_g/\delta\gamma^i_{jk} = 0 \tag{2.15}$$

arise under the gauge (2.14) as it must be in the framework of the first-order variational formalism. In general, these constraints redefine the set of equations (2.11). The problem of non-trivial consistency of (2.11) and (2.15) was considered earlier (Nikolaenko 1976, 1977a, b, 1980). In particular, it was proved there that our version does not contradict the basic experimental tests of the Einstein theory and coincides with its theoretical predictions in some cases of physical interest. It follows from these works that equations (2.15), (2.11) and (2.14) have the form

$$\lambda_0\nabla^a R_{aijk} = 0 \tag{2.16}$$

$$R_{ik} - \tfrac{1}{2}g_{ik}R + 3\lambda\eta^2 g_{ik} + \lambda_0/\Lambda_1(2R_{il}R^l_k + 2R_{ijkl}R^{jl} - g_{ik}R_{jl}R^{jl}) = 0. \tag{2.17}$$

Here λ_0 and Λ_1 are new coupling constants.

3. Axial vacuum symmetry

Let us consider the gravitational fields with axial symmetry for the case of the symmetric vacuum state in the limits:

$$\eta \to 0 \qquad R \to 0. \tag{3.1}$$

Then the field equations (2.16) and (2.17) are trivially satisfied if the vacuum Einstein equation

$$R_{ik} = 0 \tag{3.2}$$

is fulfilled.

Since the gravitational model of the vacuum is simplest in the Einstein theory we exclude those solutions of (2.16) and (2.17) which are not solutions of the vacuum equation (3.2). We shall prove that this natural assumption cancels all the axially symmetric solutions of the field equations (2.16) and (2.17).

The most general metric for the space–time with axial symmetry is given by the form (Synge 1963)

$$\mathrm{d}S^2 = \alpha^2(\mathrm{d}r^2 + \mathrm{d}z^2) + r^2\gamma^{-2}\,\mathrm{d}\varphi^2 - \gamma^2\,\mathrm{d}t^2 \tag{3.3}$$

where α and γ are arbitrary functions in variables r and z. Put $r = x^1$, $z = x^2$, $\varphi = x^3$, $t = x^4$. Then the orthonormalisable tetrad corresponding to the metric (3.3) is

$$\begin{aligned} &\theta^1 = \alpha\,\mathrm{d}x^1 \qquad &&\theta^2 = \alpha\,\mathrm{d}x^2 \\ &\theta^3 = r\gamma^{-1}\,\mathrm{d}x^3 \qquad &&\theta^4 = \gamma\,\mathrm{d}x^4. \end{aligned} \tag{3.4}$$

Applying the formula for the tensorial curvature two-form Ω_{ij} as follows

$$\Omega_{ij} = \mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega^k_j \tag{3.5}$$

and denoting

$$\begin{aligned} &\theta^1 \wedge \theta^2 = P \qquad &&\theta^3 \wedge \theta^1 = \tilde{X} \\ &\theta^4 \wedge \theta^1 = X' \qquad &&\theta^3 \wedge \theta^2 = \tilde{Y} \\ &\theta^4 \wedge \theta^2 = Y' \qquad &&\theta^3 \wedge \theta^4 = \tilde{Q} \end{aligned} \tag{3.6}$$

for the numeral-valued curvature form Ω one obtains

$$\begin{aligned} \Omega = {} & a_1(P \vee P) + a_2(\tilde{X} \vee \tilde{X}) + a_3(\tilde{Y} \vee \tilde{Y}) + a_4(X' \vee X') \\ & + a_5(Y' \vee Y') + a_6(\tilde{Q} \vee \tilde{Q}) + a_7(\tilde{X} \vee \tilde{Y}) + a_8(X' \vee Y'). \end{aligned} \tag{3.7}$$

Here $\wedge$ and $\vee$ are the exterior and symmetrised products, respectively. The values of the coefficients $a_1, a_2, \ldots, a_8$ are written in the appendix (A1).

Substituting (3.7) into the equation (2.17), considered in the limits (3.1), one obtains the set of five algebraic equations in the coefficients $a_1, a_2, \ldots, a_8$ which are denoted in the appendix (A2). To study this set it is necessary to use the constraints (2.16). It is enough to apply an algebraic consequence of the equations (2.16) which it is easy to obtain rewriting (2.16) in the form

$$\lambda_0 \nabla^a R_{aijk} = \lambda_0(\nabla_j R_{ki} - \nabla_k R_{ji}) = 0$$

and then covariantly differentiating it and using the differential Ricci identity. One obtains, finally,

$$R^{a}_{k(iq}R_{j)a}=0. \tag{3.8}$$

Here the brackets denote the cyclic permutation of indices.

Substituting (3.7) into the equation (3.8), one obtains the set of algebraic equations denoted in the appendix as (A3). These equations are additional to the set (A2).

The investigation of the set (A3) leads to the equality

$$a_7=a_8.$$

Applying formulae (A1) here, one has that

$$\partial\alpha/\partial z=0 \qquad \alpha=\alpha(r). \tag{3.9}$$

Further investigation of the sets (A2), (A3) and formulae (A1), where the restriction (3.9) is used, shows that equations (2.16) and (2.17) are not consistent in the limits (3.1). Thus, the field equations do not have axially symmetric solutions in the case of a symmetric vacuum state.

4. Conclusion

We have investigated some aspects of the problem of stable-state determination in the unified gauge theories involving gravitation. The gravitational fields were considered at a classical level. It was suggested that the gravitational Lagrangian is Weyl invariant. The conformal invariance was chosen as a main principle for construction of the Higgs form.

The theorem about the Z-type metastability free vacuum was proved. The gauge (2.14) which is equivalent to vanishing torsion and the vacuum Einstein equations were used. These are the most general restrictions on symmetries of the ground vacuum state. For asymmetric vacuum states when these conditions are not satisfied the problem of metastable vacuum existence is more complicated. A detailed discussion of this question will be presented elsewhere.

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Appendix

1.

$$\begin{aligned}
a_1&=-\tfrac{1}{2}\alpha^{-3}(\alpha_{11}+\alpha_{22})+\alpha^{-4}(\alpha_1^2+\alpha_2^2)\\
a_2&=-\tfrac{1}{2}\alpha^{-1}(\alpha^{-1}r^{-1}-\alpha^{-1}\gamma^{-1}\gamma_1)_{,1}+\tfrac{1}{2}\alpha_2\alpha^{-3}\gamma^{-1}\gamma_2\\
a_3&=\tfrac{1}{2}\alpha^{-1}(\gamma^{-1}\gamma_2\alpha^{-1})_{,2}-\tfrac{1}{2}\alpha_1\alpha^{-3}r^{-1}(1-r\gamma^{-1}\gamma_1)\\
a_4&=\tfrac{1}{2}\alpha^{-1}(\gamma_1\alpha^{-1}\gamma^{-1})_{,1}+\tfrac{1}{2}\alpha_2\gamma_2\alpha^{-3}\gamma^{-1}\\
a_5&=\tfrac{1}{2}\alpha^{-1}(\gamma_2\alpha^{-1}\gamma^{-1})_{,2}+\tfrac{1}{2}\alpha_1\gamma_1\alpha^{-3}\gamma^{-1}
\end{aligned} \tag{A1}$$

$$a_6=\tfrac{1}{2}\alpha^{-2}r^{-1}(\gamma_1-r\gamma^{-1}\gamma_1^2-r\gamma^{-1}\gamma_2^2)$$
$$a_7=-\alpha^{-1}(\alpha^{-1}r^{-1}-\gamma^{-1}\alpha^{-1}\gamma_1)_{,2}-\alpha_1\gamma^{-1}\gamma_2\alpha^{-3}$$
$$a_8=\alpha^{-1}(\gamma_1\alpha^{-1}\gamma^{-1})_{,2}-\alpha_1\gamma_2\alpha^{-3}\gamma^{-1}$$

where

$$\alpha_i=\frac{\partial\alpha}{\partial x^i}\qquad \gamma_i=\frac{\partial\gamma}{\partial x^i}$$
$$\alpha_{ij}=\frac{\partial^2\alpha}{\partial x^j\,\partial x^i}\qquad \gamma_{ij}=\frac{\partial^2\gamma}{\partial x^j\,\partial x^i}.$$

2. Using the equation $R=0$ which is equivalent to the relation $a_1+a_2+a_3-a_4-a_5-a_6=0$ one redenotes the coefficients:

$$\begin{aligned} &a_1=a_4=b_1 && a_2=a_5=b_2\\ &a_3=a_6=b_3 && a_1=a_5=b_4\\ &a_1=a_6=b_5 && a_7=b_6,\ a_8=b_7. \end{aligned} \tag{A2}$$

Equations (2.17) are equivalent to the following set of algebraic equations:

$$b_1+b_3-b_2+b_5+8{:=}(b_1b_4+b_2b_4+b_4b_5+b_1b_5-b_3b_5+b_4^2)=0$$
$$b_2+b_4+b_5+8{:=}(-b_2b_4+b_1b_4+b_1b_5+b_4b_5+b_1^2-b_3^2)=0$$
$$b_2-b_5+2{:=}(-4b_1^2+4b_3^2+b_6^2-b_7^2-4b_1b_5+4b_3b_5+4b_2b_4-4b_4b_5-4b_1b_4-8b_2b_4)=0$$
$$b_1+b_2+b_3+b_4+b_5+2{:=}(-4b_2b_4+4b_1b_4+4b_1b_5-4b_3b_5+4b_4b_5-b_7^2+b_6^2)=0$$
$$b_6-b_7+8{:=}(b_3b_6+b_1b_7+b_2b_6+b_2b_7+b_4b_7)=0$$

3.
$$\begin{aligned} &(b_1-b_4)(b_6-b_7)=0\\ &b_4(b_6-b_7)-b_7(b_1+b_3)=0\\ &b_7(b_1+b_3-2b_2)=0\\ &b_7(2b_2+b_4)=0\\ &(b_1-b_2)(b_6-b_7)=0\\ &(b_2-b_3)(b_6-b_7)=0\\ &b_3b_7+b_1b_6=0\\ &b_6(b_1+b_3-2b_2)=0\\ &b_6(2b_2+b_4)=0. \end{aligned} \tag{A3}$$

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